数理解析研究所講究録
第 679 巻 1989 年 117-126

On Higher Differentiability and Partial Regularity of the Minimizers in the Calculus of Variations

KAZUHIRO HORIHATA 堀畑 和弘
Keio University

1.Introduction

In this paper we shall treat with the following problem in the calculus of variations : Let n and N be positive integers and suppose that $\Omega \subset R^n$ is a bounded domain with the C^2 -class boundary. Then we consider the functional,

$$I[v] \equiv \sum_{\alpha,\beta=1}^{n} \sum_{i,j=1}^{N} \int_{\Omega} a_{i,j}^{\alpha,\beta}(x,v) D_\alpha v^i D_\beta v^j dx \qquad \text{for } v : \Omega \mapsto R^N , \tag{1.1}$$

where $D_\alpha v^i = \frac{\partial v^i}{\partial x_\alpha}$ $(\alpha = 1, \cdots, n \ , i = 1, \cdots, N)$ and $a_{i,j}^{\alpha,\beta}$ $(\alpha,\beta = 1, \cdots n \, , i, j = 1, \cdots, N)$ are continuously differentiable functions in $\Omega \times R^N$ satisfying the following : There exist positive numbers λ and Λ $(0 < \lambda \leq \Lambda < +\infty)$ such that $a_{i,j}^{\alpha,\beta}$ $(\alpha, \beta = 1, \cdots, n \, , i, j = 1, \cdots, N)$ satisfy for $^\forall(x,v) \in \Omega \times R^N$

$$\lambda|\zeta|^2 \leq \sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{n} a_{i,j}^{\alpha,\beta}(x\, ,v)\zeta_\alpha^i \zeta_\beta^j \leq \Lambda|\zeta|^2 \qquad \text{for } \ ^\forall(x,v) \in R^{n\times N} \text{ and } ^\forall\zeta \in R^{n\times N} , \tag{1.2}$$

$$a_{i,j}^{\alpha,\beta} = a_{j,i}^{\beta,\alpha} . \tag{1.3}$$

In addition , since the coefficients $a_{i,j}^{\alpha,\beta}$ belong to $C^1(\Omega \times R^N ; R)$, for positive numbers K_1 and K_2 there exists a positive number $L(K_1, K_2)$ such that

$$\begin{aligned} &\max_{\substack{1\leq i,j\leq N \\ 1\leq\alpha,\beta\leq n}} \max_{\substack{|x|\leq K_1 \\ |z|\leq K_2}} |a_{i,j}^{\alpha,\beta}(x,z)| + \max_{\substack{1\leq i,j\leq N \\ 1\leq\alpha,\beta\leq n}} \max_{\substack{|x|\leq K_1 \\ |z|\leq K_2}} |\frac{\partial a_{i,j}^{\alpha,\beta}}{\partial e}(x,z)| \\ &+ \max_{\substack{1\leq i,j,k\leq N \\ 1\leq\alpha,\beta\leq n}} \max_{\substack{|x|\leq K_1 \\ |z|\leq K_2}} |a_{i,j,k}^{\alpha,\beta}(x,z)| \leq L(K_1,K_2) \end{aligned} \tag{1.4}$$

$$\text{where} \quad a_{i,j,k}^{\alpha,\beta}(x,z) \equiv \frac{\partial a_{i,j}^{\alpha,\beta}}{\partial z_k}(x,z)$$

and

$$\frac{\partial a_{i,j}^{\alpha,\beta}}{\partial e}(x,z) \text{denotes the derivative in a direction of a vector } e \text{ in } R^n .$$

This implies the existence of at least a minimizer of the functional I in the Sobolev space $H^{1,2}(\Omega\,;R^N)$ and I is lower semicontinuous with respect to the weak topology of $H^{1,2}(\Omega\,;R^N)$ (see [Mo]) under an appropriate boundary condition .

First , we show that the first-derivatives of minimizers satisfies a modulus of uniform continuity in the norm $L^2_{loc}(\Omega\,;R^N)$.

Secondly , we mention a convergence theorem and a partial regularity result of the weak differentials of minimizers. However , we remark that the former theorem was proved in [Gm] , [HKL] and [Mm] .

We use the summation convention that Latin indices run from *1* to N and Greek indices run from *1* to n .

We conclude this introduction by recalling other notational conventions :

$$B_R(x_0) \equiv \{x \in R^N \, : \, |x - x_0| < R\}. \tag{1.5}$$

For a set $A \subset R^N$, we denote by $mesA$ and $|A|$ the $n-$dimensional Lebesgue measure of A .

For $u \in L^1(B_R(x_0)\,;R^N)$, we define

$$u_{x_0,R} = \frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} u(x)dx . \tag{1.6}$$

For a sufficiently small number d , we define an open set

$$\Omega_d = \Omega - \{x \in \Omega \, : \, dist(x, \partial\Omega) \le d\}, \tag{1.7}$$

where $dist(x,\partial\Omega)$ means the Euclidean metric between x and $\partial\Omega$.

For a set A in R^n , $\mathsf{H}^{(k)}(A)$ denotes the $k-$ dimensional Hausdorff measure of A (for the definition , see [Gm]) .

$e_i\,(i = 1, \cdots, n)$ means the unit vector in R^N parallel to the x_i- axis. We define a translate operator Δ_m $(m = 1, 2, \cdots, n)$ by

$$(\Delta_m f)(x) = f(x + he_m) - f(x) \qquad \text{for} \qquad f \in L^p(\Omega\,;R^N). \tag{1.8}$$

2. Main Result

Under the above preparations , we can describe

THEOREM 1. *Let u be a minimizer of the functional I in $H^{1,2}(\Omega\,;R^N)$ and let us suppose that u is a bounded , namely there exists some positive constant M such that $ess.sup|u| \le M$. Then , for any fixed domain $\tilde{\Omega}$ compactly contained in Ω , there exists positive number $\alpha = \alpha\,(n, N, \lambda, \Lambda, M, L)$ $(0 < \alpha \le 1)$ and $C = C(n, N, \lambda, \Lambda, \tilde{\Omega}, \Omega, M, L)$ such that for $h > 0$ with $h \le dist(\tilde{\Omega}\,;\partial\Omega)$ u satisfies*

$$\int_{\tilde{\Omega}} |\Delta_m(\nabla u(x))|^2 dx \le C \cdot h^{\alpha} \qquad \text{for} \qquad {}^{\forall} m\,(m = 1, 2, \cdots, n)$$

THEOREM 2. *Suppose that* $\{u_i\}_{i\geq 1}$ *is a sequence of minimizers of* I *in the space* $H^{1,2}(\Omega; R^N)$ *such that* $\{u_i\}_{i\geq 1}$ *converges strongly to a function* u_0 *in* $L^2_{loc}(\Omega; R^N)$. *Then the function* u_0 *belongs to* $H^{1,2}_{loc}(\Omega; R^N)$ *and moreover a suitable subsequence of* $\{u_i\}_{i\geq 1}$ *converges strongly to* u_0 *in* $H^{1,2}_{loc}(\Omega; R^N)$.

THEOREM 3. *Let* u *be a minimizer of the functional* I *in* $H^{1,2}(\Omega; R^N)$. *Then , for a singular set defined by*

$$S = \{x \in \Omega : \nexists \lim_{\varrho \to +0} |(Du)_{x,\varrho}|\} \cup \{x \in \Omega : \lim_{\varrho \to +0} |(Du)_{x,\varrho}| = +\infty\} \tag{2.1}$$

the following

$$H^{(\beta)}(S) = 0 \tag{2.2}$$

holds for any positive number β *satisfying* $n - 2\alpha < \beta < n$.

Remark. In the following proof , the letter C_i $(i = 1, \cdots, 14)$ means a various constant depending only on $n, N, \lambda, \Lambda, \Omega, \tilde{\Omega}, M$ and L.

PROOF OF THEOREM 1

First , a minimizer u is a weak solution of *the Euler-Lagrange* equations of the functional I , u satisfies

$$\begin{aligned} &2\int_\Omega a^{\alpha,\beta}_{i,j}(u(x),x)D_\alpha u^i(x)D_\beta\phi^j(x)\,dx \\ &+ \int_\Omega a^{\alpha,\beta}_{i,j,k}(u(x),x)D_\alpha u^i(x)D_\beta u^j(x)\phi^k(x)\,dx = 0 \end{aligned} \tag{2.3}$$

$$\text{for} \quad \forall\phi(x) \in \overset{\circ}{H}{}^{1,2}(\Omega; R^N).$$

Next, let δ be a positive number satisfying $\delta < \frac{1}{8}dist(\tilde{\Omega}, \partial\Omega)$. For each number h $(0 < h < \delta)$, the parallel transition along with x_m- axis ($m = 1, \cdots, n$) leads to

$$\begin{aligned} &2\int_\Omega a^{\alpha,\beta}_{i,j}(u(x+he_m),x+he_m)D_\alpha u^i(x+he_m)D_\beta\phi^j(x)\,dx \\ &+ \int_\Omega a^{\alpha,\beta}_{i,j,k}(u(x+he_m),x+he_m)D_\alpha u^i(x+he_m)D_\beta u^j(x+he_m)\phi^k(x)\,dx = 0 \end{aligned} \tag{2.4}$$

$$\text{for} \quad {}^\forall\phi(x) \in \overset{\circ}{C}{}^\infty(\Omega_\delta; R^N).$$

Then we have

$$\begin{aligned} &2\int_\Omega a^{\alpha,\beta}_{i,j}(u(x),x)D_\alpha\Delta_m u^i(x)D_\beta\phi^j(x)\,dx \\ &= 2\int_\Omega [a^{\alpha,\beta}_{i,j}(x,u(x)) - a^{\alpha,\beta}_{i,j}(u(x+he_m),x+he_m)]D_\alpha u^i(x+he_m)D_\beta\phi^j(x)\,dx \\ &- \int_\Omega a^{\alpha,\beta}_{i,j,k}(x+he_m,u(x+he_m))D_\alpha u^i(x+he_m)D_\beta u^j(x+he_m)\phi^k(x)dx \\ &+ \int_\Omega a^{\alpha,\beta}_{i,j,k}(x,u(x))D_\alpha u^i(x)D_\beta u^j(x)\phi^k(x)dx \end{aligned} \tag{2.5}$$

after subtracting (2.4) from (2.3) .

We now substitute $\Delta_m u(x)\zeta^2(x)$ into $\phi(x)$ in (2.5) , where $\zeta(x) \in \overset{\circ}{C}^\infty(\Omega_\delta ; R)$ is defined by

$$\zeta(x) = \begin{cases} 1 & : \Omega_{4\delta} \\ 0 & : \Omega / \Omega_{3\delta} \end{cases} \qquad \text{with} \qquad |D\zeta(x)| \le \frac{2}{\delta}. \tag{2.6}$$

Then we have

$$\begin{aligned}
&2 \int_\Omega a_{i,j}^{\alpha,\beta}(u(x),x) D_\alpha(\Delta_m u^i(x)) D_\beta(\Delta_m u^j(x))\, \zeta^2(x) dx \\
&+4 \int_\Omega a_{i,j}^{\alpha,\beta}(u(x),x) D_\alpha(\Delta_m u^i(x)) D_\beta \zeta(x) \Delta_m u^j(x) \zeta(x)\, dx \\
&\quad = 2 \int_\Omega [a_{i,j}^{\alpha,\beta}(x,u(x)) - a_{i,j}^{\alpha,\beta}(u(x+he_m), x+he_m)] D_\alpha u^i(x+he_m) \\
&\qquad [D_\beta(\Delta_m u^j(x))\zeta^2(x) + 2\Delta_m u^j(x) D_\beta\zeta(x)\zeta(x)]dx \\
&- \int_\Omega a_{i,j,k}^{\alpha,\beta}(x+he_m, u(x+he_m)) D_\alpha u^i(x+he_m) D_\beta u^j(x+he_m) \Delta_m u^k(x) \zeta^2(x) dx \\
&+ \int_\Omega a_{i,j,k}^{\alpha,\beta}(x,u(x)) D_\alpha u^i(x) D_\beta u^j(x) \Delta_m u^k(x) \zeta^2(x) dx
\end{aligned} \tag{2.7}$$

Here, we estimate the left-hand side of (2.7) , which we call (L) , from below . First , by using (1.2) , we have

$$\begin{aligned}
(L) \ge\, & 2\lambda \int_\Omega |D(\Delta_m u(x))|^2 \zeta^2(x) dx \\
& - 4nN\Lambda \int_\Omega |D(\Delta_m u(x))| |\zeta(x)| \Delta_m u(x)| |D\zeta(x)| dx .
\end{aligned} \tag{2.8}$$

Second , applying *the Schwarz inequality* to the second term of (2.8) with $\varepsilon = \frac{\lambda}{2nN\Lambda}$, we have

$$\begin{aligned}
(L) \ge\, & 2\lambda \int_\Omega |\Delta_m(Du(x))|^2 \zeta^2(x) dx \\
& - 2\varepsilon nN\Lambda \int_\Omega |\Delta_m(Du(x))|^2 \zeta^2(x) dx \\
& - \frac{2nN\Lambda}{\varepsilon} \int_\Omega |D\zeta(x)|^2 |\Delta_m u(x)|^2 dx \\
\ge\, & \lambda \int_\Omega |\Delta_m(Du(x))|^2 \zeta^2(x) dx \\
& - 2\frac{2(nN\Lambda)^2}{\lambda} \int_\Omega |D\zeta^2(x)| |\Delta_m u(x)|^2 dx .
\end{aligned} \tag{2.9}$$

On the other hand , we perform the estimates of the right-hand side of (2.7) , which we call

(R) .

$$(R) = -2\int_\Omega \int_0^1 \frac{da_{i,j}^{\alpha,\beta}}{dt}(x+the_m\,,\, u(x) + t\Delta_m u(x))dt\, D_\alpha u^i(x+he_m)$$
$$[D_\beta(\Delta_m u^j(x))\zeta^2(x) + 2\Delta_m u^j(x) D_\beta\zeta(x)\zeta(x)\,]dx$$
$$- \int_\Omega a_{i,j,k}^{\alpha,\beta}(x+he_m, u(x+he_m))D_\alpha u^i(x+he_m)D_\beta u^j(x+he_m)\Delta_m u^k(x)\zeta^2(x)dx$$
$$+ \int_\Omega a_{i,j,k}^{\alpha,\beta}(x,u(x))D_\alpha u^i(x)D_\beta u^j(x)\Delta_m u^k(x)\zeta^2(x)dx \tag{2.10}$$

By using (1.4) and the boundedness of u , and applying *the Schwarz inequality* to (2.10) , we have

$$(R) \le C_1 \int_\Omega (h + |\Delta_m u(x)|)\,\zeta(x)dx\,.$$
$$+C_1 \int_\Omega (h + |\Delta_m u(x)| + |\Delta_m u(x)|^2)[\,|Du(x+he_m)|^2 + |Du(x)|^2\,]\zeta(x)^2\,dx\,. \tag{2.11}$$

Thus , by combining (2.9) with (2.11) , we have the following :

$$\int_{\Omega_{4\delta}} |\Delta_m(Du(x))|^2 dx \le C_2 \int_{\Omega_{3\delta}} |\Delta_m u(x)|^2 dx$$
$$+C_2 \int_{\Omega_{3\delta}} (h + |\Delta_m u(x)|)dx$$
$$+C_2 \int_{\Omega_{3\delta}} (h + |\Delta_m u(x)|)[|Du(x)|^2 dx + |Du(x+he_m)|^2]dx \tag{2.12}$$

Here , it is a well-known fact that a minimizer u satisfies *a so-called Caccioppoli inequality* (see [Gm]) : There exists a positive constant C , depending only on n , N , λ, Λ , M such that

$$\int_{B_R} |Du(x)|^2 dx \le \frac{C}{R^2}\int_{B_{2R}} |u(x) - u_R|^2 dx \tag{2.13}$$

holds for any ball $B_{2R} \subset\subset \Omega$ with $0 < {}^\forall R < \delta$. A direct application of the above inequality to *Gering inequality* due to *F.W.Gering* [Ge] (see also [Gm]) leads to the following: There exists a positive number $p\,(p > 2)$, which can be supposed to satisfy $p < 4$ and C depending only on n , N , λ , Λ , Ω , M such that $Du(x)$ belongs to $L^p_{loc}(\Omega\,; R^N)$ and moreover

$$\left(\frac{1}{|\tilde\Omega|}\int_{\tilde\Omega} |Du(x)|^p dx\right)^{\frac{1}{p}} \le \left(\frac{1}{|\Omega|}\int_\Omega |Du(x)|^2 dx\right)^{\frac{1}{2}} \tag{2.14}$$

holds for ${}^\forall\tilde\Omega \subset\subset \Omega$.

Thus , we apply *Hölder inequality* to the second term of the right-hand side of (2.12) and we have

$$\int_{\Omega_{4\delta}} |\Delta_m(Du(x))|^2 dx \le C_4 \int_{\Omega_{2\delta}} [h + |\Delta_m u(x)| + |\Delta_m u(x)|^2]dx$$

$$+C_4 \left[\int_{\Omega_{2\delta}} |D_m u(x)|^p dx\right]^{\frac{2}{p}} \left[\int_{\Omega_{2\delta}} |\Delta_m u(x)|^{\frac{p}{p-2}} dx\right]^{\frac{p-2}{p}} . \tag{2.15}$$

In addition , by using (2.13),(2.14) and the boundedness of u , we have

$$\int_{\Omega_{4\delta}} |\Delta_m(Du(x))|^2 dx \leq C_4 \int_{\Omega_{2\delta}} [h + |\Delta_m u(x)| + |\Delta_m u(x)|^2] dx$$

$$+ C_5 \left[\int_{\Omega_{2\delta}} |\Delta_m u(x)|^{\frac{p}{p-2}} dx \right]^{\frac{p-2}{p}} . \tag{2.16}$$

Since $2 < p < 4$ implies $\frac{p}{p-2} > 2$ it follows from the boundedness of $u(x)$ that

$$\int_{\Omega_{4\delta}} |\Delta_m(Du(x))|^2 dx \leq C_4 \int_{\Omega_{2\delta}} [h + |\Delta_m u(x)| + |\Delta_m u(x)|^2] dx$$

$$+ C_6 \left[\int_{\Omega_{2\delta}} |\Delta_m u(x)|^2 dx \right]^{\frac{p-2}{p}} . \tag{2.17}$$

Also , from *Newton-Leibnitz formula* and *a Caccioppoli inequality* , we obtain

$$\int_{\Omega_{2\delta}} |\Delta_m u(x)|^2 dx \leq C_7 h^2 \int_{\Omega_{\delta}} |Du(x)|^2 dx \leq C_8 h^2 . \tag{2.18}$$

Consequently , from (2.17) and (2.18) , we deduce

$$\int_{\Omega_{4\delta}} |\Delta_m(Du(x))|^2 dx \leq C_8 h^{\frac{2}{p}(p-2)} . \tag{2.19}$$

[1] Also , for any fixed unit vector e one can easily prove

$$\int_{\Omega_{4\delta}} |\Delta_e(Du(x))|^2 dx \leq C_8 h^{\frac{2}{p}(p-2)} \tag{2.20}$$

Proof of Theorem 2

From (2.19) , we obtain an equi-continuity of a sequence of minimizers in $H^{1,2}_{loc}(\Omega\,; R^N)$. Also , it follows from (2.14) that a sequence of minimizers satisfies a uniform boundedness in $H^{1,2}_{loc}(\Omega\,; R^N)$. Thus we obtain the assertion of this theorem from *Rellich - Kondrachev theorem* , (see [Ad]) .

Proof of Theorem 3

The proof of this Theorem is based on estimate (2.20) and the following lemma due to [Gi] (see also [Gm]) .

[1] The estimate (2.19) and (2.20) play an important role in the proofs of the Theorem 2 and Theorem 3

LEMMA 3.1.

Let v be a function in $L^1_{loc}(\Omega)$ and β be any number satisfying $n - 2\alpha < \beta < n$. Set

$$E_\beta = \{x \in \Omega : \limsup_{\rho \to +0} \rho^{-\beta} \int_{B_\rho(x)} |v(y)|\, dy > 0\}. \tag{2.21}$$

Then , we have

$$H^{(\beta)}(E_\beta) = 0. \tag{2.22}$$

First, to apply Lemma 3.1 to the proof of Theorem 3 we construct a support function defined as follows : For $\rho_k = \delta(\frac{1}{2})^{k+1}$ $(k = 1, 2, \cdots)$ with $\delta = dist(\tilde{\Omega}, \partial\Omega)$ and a sequence $\{e_k\}_{k \geq 1}$ of unit vectors in R^n we define

$$\varphi_k(y) = \rho_k^{-(n-\beta)-\varepsilon} |Du(y + \rho_k e_k) - Du(y)|^2 \quad \text{with} \quad \varepsilon = \frac{1}{2}(2\alpha - (n - \beta)). \tag{2.23}$$

When we set

$$\phi_k(y) = \sum_{j=1}^{k} \varphi_j(y), \tag{2.24}$$

one easily finds that the function $\phi_k(y)$ is a non-decreasing function for k and the following

$$\int_{\tilde{\Omega}} \phi_k(y) dy = \sum_{j=1}^{k} \int_{\tilde{\Omega}} \varphi_j(y) dy$$

$$= \sum_{j=1}^{k} \rho_j^{-(n-\beta)-\varepsilon} \int_{\tilde{\Omega}} |Du(y + \rho_k e_k) - Du(y)|^2 dy \leq C_8 \sum_{j=1}^{k} \rho_j^{2\alpha-(n-\beta)-\varepsilon} \tag{2.25}$$

$$\leq C_8 \sum_{j=1}^{k} \delta^{\frac{1}{2}(2\alpha-(n-\beta))} (\frac{1}{2})^{\frac{j}{2}\{2\alpha-(n-\beta)\}} = C_8 \delta^{\frac{1}{2}(2\alpha-(n-\beta))} \sum_{j=1}^{k} 2^{-\frac{j}{2}\{2\alpha-(n-\beta)\}} \leq C_9 < \infty.$$

follows from (2.20) and assumption of $2\alpha - (n - \beta) = \beta - (n - 2\alpha) > 0$.

Thus $\{\phi_k\}_{k \geq 1}$ is a sequence of measurable functions and moreover , putting $\phi_\infty(y) = \lim_{k\to\infty} \phi_k(y)$, we obtain from *Beppo-Levi Theorem*

$$\int_{\tilde{\Omega}} \phi_\infty(y) dy = \lim_{k \to \infty} \int_{\tilde{\Omega}} \phi_k(y) dy \leq C_{10}. \tag{2.26}$$

Consequently , $\phi_\infty(y)$ is an integrable function on $\tilde{\Omega}$ and

$$\varphi_k(y) \leq \phi_\infty(y) \quad \text{for any } k \quad \text{and almost all} \quad y \in \tilde{\Omega}. \tag{2.27}$$

To complete the proof of theorem , it is sufficient to show

$$S \subset E_\beta \quad \text{, namely,} \quad \text{if} \quad x_0 \notin E_\beta, \quad \text{then} \quad x_0 \notin S. \tag{2.28}$$

Now we fix $x_0 \notin E_\beta$, Then we show that the function

$$r \longmapsto (Du)_{x_0,r} = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} (Du)(y)dy \tag{2.29}$$

is a continuous and bounded function in the open interval $(0, \delta)$ with $\delta = dist(x_0, \partial\Omega)$.

At first , we shall estimate $|(Du)_{x_0,R_i} - (Du)_{x_0,R_{i+1}}|$ $(i = 1, 2, \cdots)$. Also, by integrating the following (2.30) over $B_{R_i}(x_0)$ $R_i = \frac{\delta}{2}(\frac{1}{2})^2$ $(i = 1, , \cdots)$,

$$\begin{aligned} &|(Du)_{x_0,R_i} - (Du)_{x_0,R_{i+1}}| \\ &\leq |(Du)_{x_0,R_i} - (Du)(x)| + |(Du)_{x_0,R_{i+1}} - (Du)(x)|\,. \end{aligned} \tag{2.30}$$

we obtain

$$\begin{aligned} &|B_{R_i}| \cdot |Du_{x_0,R_i} - Du_{x_0,R_{i+1}}| \\ &\leq \int_{B_{R_i}} |Du_{x_0,R_i} - Du(x)|dx + \int_{B_{R_i}} |Du_{x_0,R_{i+1}} - Du(x)|dx\,. \end{aligned} \tag{2.31}$$

Next, dividing (2.31) by $|B_{R_i}|$ and by using Hölder inequality , we have

(2.32)

$$\begin{aligned}
&|Du_{x_0,R_i} - Du_{x_0,R_{i+1}}| \\
&\leq \frac{1}{|B_{R_i}|} \int_{B_{R_i}} |Du_{x_0,R_i} - Du(x)|dx + \frac{1}{|B_{R_i}|} \int_{B_{R_i}} |Du_{x_0,R_{i+1}} - Du(x)|dx \\
&\leq \frac{1}{|B_{R_i}|} \int_{B_{R_i}} |\frac{1}{|B_{R_i}|} \int_{B_{R_i}} Du(y)\,dy - Du(x)|dx + \frac{1}{|B_{R_i}|} \int_{B_{R_i}} |\frac{1}{|B_{R_{i+1}}|} \int_{B_{R_{i+1}}} Du(y)\,dy - Du(x)|dx \\
&\leq \frac{1}{|B_{R_i}|^2} \int_{B_{R_i}} dx \int_{B_{R_i}} |Du(y) - Du(x)|dy + \frac{1}{|B_{R_i}||B_{R_{i+1}}|} \int_{B_{R_i}} dx \int_{B_{R_{i+1}}} |Du(y) - Du(x)|dy \\
&\leq \frac{1+2^n}{|B_{R_i}|^2} \int_{B_{R_i}} dx \int_{B_{R_i}} |Du(y) - Du(x)|dy. \\
&\leq \frac{1+2^n}{|B_{R_i}|^2} [\int_{B_{R_i}} dx \int_{B_{R_i}} dy]^{\frac{1}{2}} [\int_{B_{R_i}} dx \int_{B_{R_i}} |Du(y) - Du(x)|^2 dy]^{\frac{1}{2}} \\
&\leq \frac{1+2^n}{|B_{R_i}|} [\int_{B_{R_i}} dx \int_{B_{R_i}} |Du(y) - Du(x)|^2 dy]^{\frac{1}{2}}\,.
\end{aligned}$$

Here, we extend $Du(x)$ to be zero outside B_{R_i} and successively rewrite it to be $Du(x)$ for convenience. Then we continue the estimates of (2.32) as follows : From the change of variables,

$$\begin{cases} \bar{x} = x, \\ \bar{y} = y - x \end{cases}$$

we obtain

$$\frac{(1+2^n)}{|B_{R_i}|}[\int_{B_{R_i}} dx \int_{B_{R_i}} |(Du)(y) - (Du)(x)|^2 dy]^{1/2} tag2.33$$

$$= \frac{(1+2^n)}{|B_{R_i}|}[\int_{B_{R_i}} dx \int_{B_{2R_i}(x)} |(Du)(y) - (Du)(x)|^2 dy]^{1/2}$$

$$= \frac{(1+2^n)}{|B_{R_i}|}[\int_{B_{R_i}} d\bar{x} \int_{B_{2R_i}(0)} |(Du)(\bar{x}+\bar{y}) - (Du)(\bar{x})|^2 dy]^{1/2}.$$

By using *Fubini Theorem* and successively the mean value theorem , there exists a vector $\bar{y}_i^* \in R^n$ with $0 < |\bar{y}_i^*| < 2R_i$ such that

$$(2.34) \qquad (2.33) = [\frac{c_{11}}{|B_{R_i}|}\int_{B_{R_i}} |(Du)(\bar{x}+\bar{y}_i^*) - (Du)(\bar{x})|^2 d\bar{x}].$$

From (2.32) and (2.34) , we obtain

$$(2.35) \qquad |(Du)_{x_0,R_i} - (Du)_{x_0,R_{i+1}}| \leq C_{12}\left[\frac{R_i^{n-\beta+\varepsilon}}{|B_{R_i}|}\int_{B_{R_i}} \frac{|(Du)(\bar{x}+\bar{y}_i^*) - (Du)(\bar{x})|^2}{|\bar{y}_i^*|^{n-\beta+\varepsilon}} d\bar{x}\right]^{\frac{1}{2}}.$$

Next we shall show that $\{Du_{x_0,r}\}$ $(r > 0)$ is a *Cauchy filter* . Let r and R $(r < R)$ be positive numbers sufficiently small and then we can take positive integer j and i $(i \leq j)$ such that $R_{j+1} < r \leq R_j$ and $R_{i+1} < R \leq R_i$. We estimate $|\,Du_{x_0,r} - Du_{x_0,R}\,|$ by dividing it into the following three terms :

$$(2.36) \qquad \begin{aligned} &|\,Du_{x_0,r} - Du_{x_0,R}\,| \\ &\leq |\,Du_{x_0,r} - Du_{x_0,R_j}\,| + |\,Du_{x_0,R_j} - Du_{x_0,R_i}\,| + |\,Du_{x_0,R} - Du_{x_0,R_i}\,| \end{aligned}$$

Thus , by the same way as above , for $0 < r < R < \delta$, the following holds :

$$(2.37) \qquad |(Du)_{x_0,r} - (Du)_{x_0,R}| \leq C_{12}\sum_{k=i}^{j} R_k^{\varepsilon/2}\left[R_k^{-\beta}\int_{B_{R_k}} \frac{|(Du)(\bar{x}+\bar{y}_i^*) - (Du)(\bar{x})|^2}{|\bar{y}_i^*|^{n-\beta+\varepsilon}} d\bar{x}\right]^{\frac{1}{2}}.$$

We obtain from $\beta - (n-2\alpha) \geq 0$,

$$\sum_{k=i}^{j} R_k^{\frac{\varepsilon}{2}} \leq R_i^{\frac{\varepsilon}{2}} \frac{1-(\frac{\delta}{2})^{(j-i)\frac{\varepsilon}{2}}}{1-(\frac{\delta}{2})^{\frac{\varepsilon}{2}}}$$

By noting

$$\frac{|(Du)(\bar{x}+\bar{y}_k^*) - (Du)(\bar{x})|^2}{|\bar{y}_k^*|^{n-\beta+\varepsilon}} \leq \phi_\infty(x) \qquad a.e \quad \bar{x} \in \tilde{\Omega} \text{ and } k = 1,2,\cdots.$$

we can continue to estimate (2.37) as follows :

$$|(Du)_{x_0,r} - (Du)_{x_0,R}|$$

$$\leq C_{14} R^{\frac{\epsilon}{2}} \left[\underset{k>0}{ess.sup} R_k^{-\beta} \int_{B_{R_k}} \phi_\infty(y) dy \right]^{\frac{1}{2}} . \tag{2.38}$$

Also , from (2.28) , there exists a constant K such that

$$|(Du)_{x_0,r} - (Du)_{x_0,R}| \leq C_{14} K^{\frac{1}{2}} R^{\frac{\beta-(n-2\alpha)}{2}} \tag{2.39}$$

This shows that $\{(Du)_{x_0,r}\}_{r>0}$ is *a Cauchy filter* . Thus, $\underset{R\to+0}{lim}(Du)_{x_0,R}$ surely exists. Also , from (2.39), we obtain

$$|(Du)_{x_0,r} - (Du)_{x_0,R}| \leq C_{14} K^{\frac{1}{2}} R^{\frac{\beta-(n-2\alpha)}{2}} . \tag{2.40}$$

Then

$$\underset{R\to+0}{lim} |(Du)_{x_0,R}| \leq |(Du)_{x_0,\delta/4}| + C_{14} K^{\frac{1}{2}} (\delta/4)^{\frac{\beta-(n-2\alpha)}{2}} . \tag{2.41}$$

Consequently , $\underset{R\to+0}{lim}(Du)_{x_0,R}$ exists and is finite. This shows $x_0 \notin S$.

References

[Ad]. Adams R A, "Sobolev space," Academic Press, 1975.

[Ge]. Gehring F W, *The L^p integrability of the partial derivatives of a quasi conformal mapping*, Acta Math 130.

[GG1]. Giaquinta M and Giusti E, *Differentiability of Minima of Non-Differentiable Functions*, Invent. math 72.

[GG2]. Giaquinta M and Giusti E, *Sharp Estimates for the Derivatives of Local Minima of Variational Integrals*, Boll U.M.I 3-A.

[Gi]. Giusti E, *Precisazione delle funzione $H^{1,p}$ e singolarita delle soluzioni deboli di sistemi ellittici non lineari*, Boll U.M.I 2.

[Gm]. Giaquinta M, "Multiple integrals in the calculus of variations and nonlinear elliptic systems," Annals of Math Studies 105 Princeton, 1983.

[Mm]. Meier M, *Liouville theorems , partial regularity and Höldercontinuity of weak solutions to quasi linear ellptic systems*, Preprint no. 535 Sonderforschungsbereich 72.

[Mo]. Morrey C B Jr, "Multiple integrals in the calculus of variations," Springer, 1966.

[Ne]. Nečas J, "Les Méthodes en theorie des équations elliptiques," Praha Akademia, 1967.